ASN-Minimax double sampling plans by variables for two-sided specification limits when σ is unknown

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Abstract: ASN-Minimax double sampling plans by variables for a normally distributed quality characteristic with unknown σ and two-sided specification limits are introduced. These plans base on the essentially Maximum-Likelihood (ML) estimator p^* and the Minimum Variance Unbiased (MVU) estimator $\hat{p}$ of the fraction defective p. The operation characteristic (OC) of the ASN-Minimax double sampling plans is determined by using the independent random variables p_1^*, p_2^* and $\hat{p}_1$, $\hat{p}_2$, which relate to the first and second samples, respectively. The maximum of the average sample number (ASN) of these plans is shown to be considerably smaller than the sample size of the corresponding single sampling plans.

Keywords: Acceptance sampling by variables, ASN-Minimax double sampling plan, essentially Maximum-Likelihood estimator, Minimum Variance Unbiased estimator

1. Introduction

For a normally distributed characteristic $X \sim N(\mu, \sigma^2)$ with unknown $\sigma > 0$ and lower and upper specification limits L and U, respectively, the fraction defective function $p(\mu, \sigma)$ is defined as

$$p(\mu, \sigma) := P(X < L) + P(X > U) = \Phi\left(\frac{L - \mu}{\sigma}\right) + \Phi\left(\frac{\mu - U}{\sigma}\right), \quad (1)$$

where Φ denotes the standard normal distribution function. Given a large-sized lot and a single sample $X_1, ..., X_n$, $(n > 3)$ with

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2,$$

there are two well-known procedures to determine single sampling plans. The

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first procedure (cf. BRUHN-SUHR/KRUMBHOLZ (1990)) is based on the essentially Maximum-Likelihood (ML) estimator

$$p^* = p(\overline{X}, S) = \Phi\left(\frac{L - \overline{X}}{S}\right) + \Phi\left(\frac{\overline{X} - U}{S}\right). \tag{2}$$

The second procedure (cf. BRUHN-SUHR/KRUMBHOLZ (1991)) is based on the Minimum Variance Unbiased (MVU) estimator $\hat{p}$ (cf. KOLMOGOROV (1953), LIEBERMAN/RESNIKOFF (1955)). Let

$$V := max\left\{0, \frac{1}{2} - \frac{1}{2}\frac{\overline{X} - L}{S}\frac{\sqrt{n}}{n - 1}\right\}, \tag{3}$$

$$W := max\left\{0, \frac{1}{2} - \frac{1}{2}\frac{U - \overline{X}}{S}\frac{\sqrt{n}}{n - 1}\right\}, \tag{4}$$

$$b(x) := \frac{\Gamma(n - 2)}{\Gamma\left(\frac{n - 2}{2}\right)\Gamma\left(\frac{n - 2}{2}\right)} x^{\frac{n}{2} - 2}(1 - x)^{\frac{n}{2} - 2} \quad (0 < x < 1), \tag{5}$$

$$B(x) := \int_0^x b(t)dt, \tag{6}$$

where Γ denotes the gamma function and $B(x)$ and $b(x)$ the distribution and density function of the symmetrical beta distribution with parameter $\frac{n - 2}{2}$. The MVU estimator $\hat{p}$ is defined by

$$\hat{p} := B(V) + B(W). \tag{7}$$

The lot is accepted within the single sampling plan

$$(n^*, \ k^*), \quad \text{if} \quad p^* \leq k^* \quad \text{for the first procedure,}$$
$$(\hat{n}, \ \hat{k}), \quad \text{if} \quad \hat{p} \leq \hat{k} \quad \text{for the second procedure.}$$

In this paper, we introduce ASN-Minimax (AM) double sampling plans for both procedures. For given acceptable quality level p_1, rejectable quality level p_2 and levels α and β of Type-I and Type-II error, respectively, the AM-double sampling plans feature minimal maximal average sample num-

ber (ASN) while satisfying the classical two-points-condition on the operation characteristic (OC). We define the double sampling plans by using in the second stage only the information obtained from the second sample. Thus, the double-sampling-plan-OC can be determined by implementing the corresponding single-sampling-plan-OC. The double sampling plans are computed in a similar way as the single sampling plans.

In the next section, we give two well-known theorems regarding the single-sampling-plan-OC for each procedure and derive with their help the corresponding double-sampling-plan-OC. The computation of the AM-double sampling plans is elucidated in the third section. In order to determine two sample sizes and three critical values of the two-sided AM-double sampling plan, we use the corresponding one-sided AM-approximation.

2. The double-sampling-plan-OC

Before deriving the double-sampling-plan-OC, we should bring together two well-known theorems from the literature on single sampling. Let

$$L_{(n^*, \ k^*)}(\mu, \sigma) = P(p^* \le k^*) \tag{8a}$$

and

$$L_{(\hat{n}, \ \hat{k})}(\mu, \sigma) = P(\hat{p} \le \hat{k}) \tag{8b}$$

denote the single-sampling-plan-OC for the first and second procedures, respectively, and let g_r be the density function of the χ^2 distribution with r degrees of freedom.

Theorem 1: It holds that (cf. Bruhn-Suhr/Krumbholz (1990)):

$$
L_{(n^*, \ k^*)}(\mu, \sigma) = \int_0^C \left\{ \Phi\left(\frac{\sqrt{n^*}}{\sigma}\left(\mu\left(\sigma\sqrt{\frac{t}{n^*-1}}, k^*\right) - \mu\right)\right) \right.
$$
$$
\left. -\Phi\left(\frac{\sqrt{n^*}}{\sigma}\left(\acute{\mu}\left(\sigma\sqrt{\frac{t}{n^*-1}}, k^*\right) - \mu\right)\right) \right\} g_{n^*-1}(t)dt \tag{9}
$$

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with

$$C = \frac{(n^* - 1)(L - U)^2}{4\sigma^2 \left(\Phi^{-1} \left(\frac{k^*}{2} \right) \right)^2} \quad \text{and} \quad \acute{\mu}(\sigma, p) = L + U - \mu(\sigma, p).$$

Theorem 2: It holds that (cf. Bruhn-Suhr/Krumbholz (1991)):

$$L_{(\hat{n}, \ \hat{k})}(\mu, \sigma) = \int_0^{A(B^{-1}(\hat{k}), 0)} \left\{ \Phi \left(-\delta_U - \sqrt{t(\hat{n} - 1)} \right) - \right.$$

$$- \Phi \left(-\delta_L + (1 - 2B^{-1}(\hat{k})) \sqrt{t(\hat{n} - 1)} \right) \right\} g_{\hat{n}-1}(t) dt +$$

$$+ 2 \int_0^{B^{-1}(\hat{k})} \left\{ \int_0^{A(\psi(y), y)} \Phi' \left(-\delta_U + (2y - 1)\sqrt{t(\hat{n} - 1)} \right) \cdot \right.$$

$$\left. \cdot \sqrt{t(\hat{n} - 1)} g_{\hat{n}-1}(t) dt \right\} dy \qquad (10)$$

with

$$A(x, y) = \begin{cases} \dfrac{\hat{n}(U - L)^2}{4\sigma^2(\hat{n} - 1)(1 - x - y)^2} & \text{if} \quad x + y < 1 \\ \infty & \text{else,} \end{cases}$$

$$\delta_U = \sqrt{\hat{n}} \frac{\mu - U}{\sigma}, \quad \delta_L = \sqrt{\hat{n}} \frac{\mu - L}{\sigma} \quad \text{and} \quad \psi(y) = B^{-1}(\hat{k} - B(y)).$$

Definition 1: The double sampling plan by variables

$$\lambda^* = \begin{pmatrix} n_1^* & k_1^* & k_2^* \\ n_2^* & k_3^* \end{pmatrix} \quad \text{for the first procedure,}$$

$$\widehat{\lambda} = \begin{pmatrix} \hat{n}_1 & \hat{k}_1 & \hat{k}_2 \\ \hat{n}_2 & \hat{k}_3 \end{pmatrix} \quad \text{for the second procedure,}$$

with $n_1^*, n_2^*, \hat{n}_1, \hat{n}_2 \in \mathbb{N}$; $n_1^*, n_2^*, \hat{n}_1, \hat{n}_2 \geq 2$; $k_1^*, k_2^*, k_3^*, \hat{k}_1, \hat{k}_2, \hat{k}_3 \in \mathbb{R}^+$; $k_1^* \leq k_2^*$, $\hat{k}_1 \leq \hat{k}_2$, is defined as follows:

(i) Observe a first sample of size n_1^* for the first procedure, $\hat{n}_1$ for the second

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procedure and compute $p_1^*, \hat{p}_1$.

If $p_1^* \le k_1^*$, $\hat{p}_1 \le \hat{k}_1$, accept the lot.
If $p_1^* > k_2^*$, $\hat{p}_1 > \hat{k}_2$, reject the lot.
If $k_1^* < p_1^* \le k_2^*$, $\hat{k}_1 < \hat{p}_1 \le \hat{k}_2$, go to (ii).

(ii) Observe a second sample of size n_2^* for the first procedure, $\hat{n}_2$ for the second procedure and compute $p_2^*, \hat{p}_2$.

If $p_2^* \le k_3^*$, $\hat{p}_2 \le \hat{k}_3$, accept the lot.
If $p_2^* > k_3^*$, $\hat{p}_2 > \hat{k}_3$, reject the lot.

Let $p^{est} \in \{p^*, \hat{p}\}$ denote the used p-estimator, $(n,\ k) \in \{(n^*,\ k^*), (\hat{n},\ \hat{k})\}$ the corresponding single sampling plan and $\lambda = \begin{pmatrix} n_1 & k_1 & k_2 \\ n_2 & k_3 \end{pmatrix}$, $\lambda \in \{\lambda^*, \widehat{\lambda}\}$ the corresponding double sampling plan given in Definition 1. Considering the independence of the samples, the OC $L_\lambda(\mu, \sigma)$ is given by

$$L_\lambda(\mu, \sigma) = P_{(\mu,\sigma)}(p_1^{est} \le k_1) + P_{(\mu,\sigma)}(p_2^{est} \le k_3,\ k_1 < p_1^{est} \le k_2)$$
$$= L_{(n_1,k_1)}(\mu, \sigma) + L_{(n_2,k_3)}(\mu, \sigma)\left(L_{(n_1,k_2)}(\mu, \sigma) - L_{(n_1,k_1)}(\mu, \sigma)\right). \quad (11)$$

The ASN $N_\lambda(\mu, \sigma)$ is given by

$$N_\lambda(\mu, \sigma) = n_1 + n_2\, P_{(\mu,\sigma)}(k_1 < p_1^{est} \le k_2) \qquad (12)$$

with

$$P_{(\mu,\sigma)}(k_1 < p_1^{est} \le k_2) = L_{(n_1,k_2)}(\mu, \sigma) - L_{(n_1,k_1)}(\mu, \sigma).$$

Depending on μ and σ, L_λ and N_λ are bands in p (Figures 1, 2). The maximum $N_{max}(\lambda)$ of $N_\lambda(\mu, \sigma)$ is for the computation of the AM-double sampling plans λ_{AM} irrelevant. However, we calculate $N_{max}(\lambda)$ as a basis for comparison with the ASN maximum of the AM-one-sided approximation, which is used in the computation of λ_{AM}. A detailed description of the computation of λ_{AM} is given in the next section.
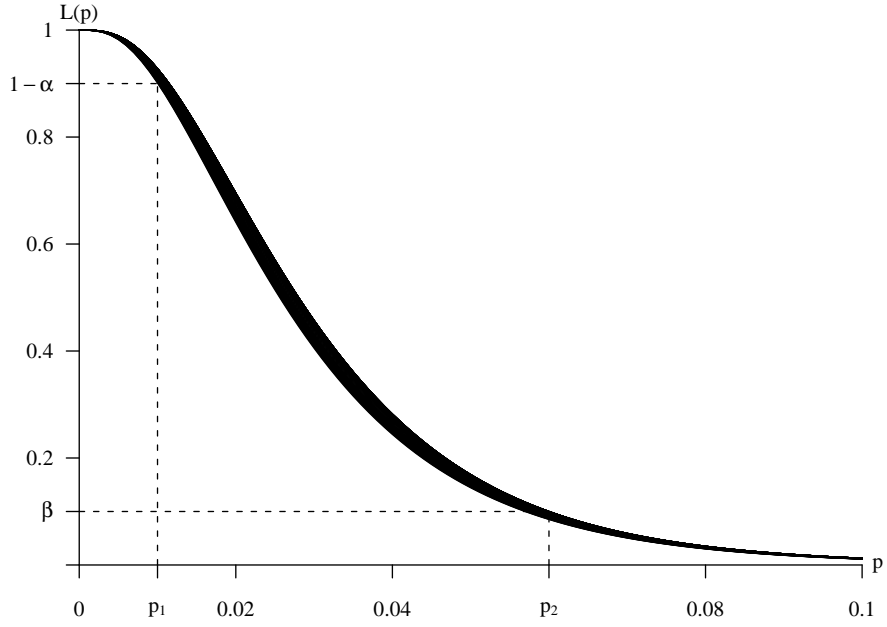
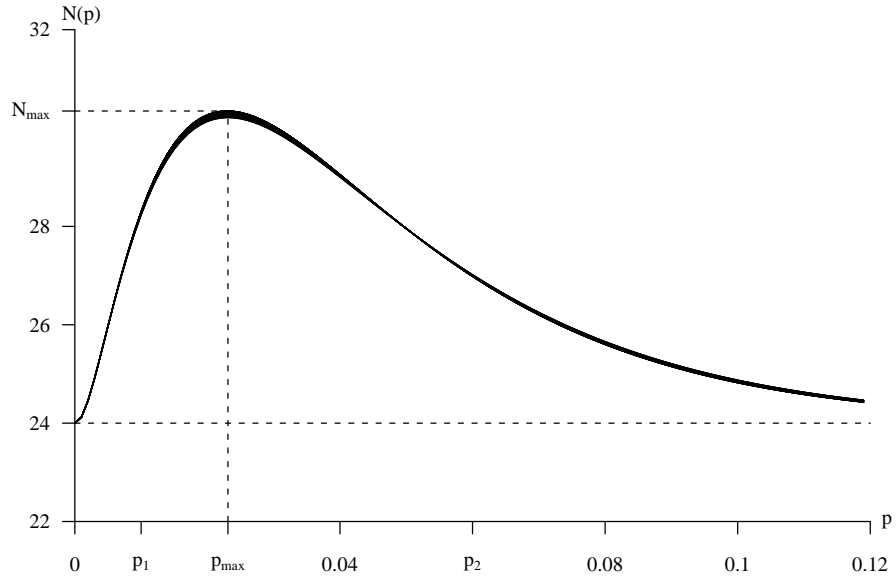Figure 1: OC-band for λ_{AM}^* defined by $p_1 = 0.01$, $p_2 = 0.06$ and $\alpha = \beta = 0.1$

Figure 2: ASN-band for $\widehat{\lambda}_{AM}$ defined by $p_1 = 0.01$, $p_2 = 0.06$ and $\alpha = \beta = 0.1$

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3. The computation of the AM-double sampling plans

The two-sided single sampling plan is calculated by using the one-sided approximation $(\tilde{n},\ \tilde{k})$, which, without loss of generality is chosen for the case of an upper specification limit U. For given acceptable quality level p_1, rejectable quality level p_2 and levels α and β of Type-I and Type-II error, respectively, the one-sided approximation $(\tilde{n},\ \tilde{k})$ is computed by fulfilling the condition

$$
\begin{aligned}
&\text{(i)} \quad F_{\tilde{n}-1,\ \delta(p_1)}(l) \geq 1 - \alpha \\
&\text{(ii)} \quad F_{\tilde{n}-1,\ \delta(p_2)}(l) \leq \beta \\
&\text{(iii)} \quad \tilde{n} \stackrel{!}{=} \min \\
&\text{(iv)} \quad \tilde{k} =
\begin{cases}
\Phi\left(\dfrac{l}{\sqrt{\tilde{n}}}\right) & \text{for the first procedure} \\
B\left(\dfrac{1}{2} + \dfrac{l}{2(\tilde{n}-1)}\right) & \text{for the second procedure.}
\end{cases}
\end{aligned}
\tag{13}
$$

$F_{\tilde{n}-1,\ \delta(p)}$ denotes the distribution function of the noncentral t-distribution with $\tilde{n}-1$ degrees of freedom and the non-centrality parameter

$$
\delta(p) = \sqrt{\tilde{n}}\ \Phi^{-1}(p). \tag{14}
$$

Let for the case of an upper specification limit U, $\phi^* = \begin{pmatrix} n_1 & l_1 & l_2 \\ n_2 & l_3 \end{pmatrix}$ denote the AM-double sampling plan defined by using the independent statistics

$$
T_i = \sqrt{n_i}\ \frac{\overline{X}_i - U}{S_i}, \quad i = 1, 2. \tag{15}
$$

The OC and ASN for a double sampling plan ϕ in this case (cf. HILBERT (2005)) are given by

$$
L_\phi(p) = F_{n_1-1,\ \delta_1(p)}(l_1) + F_{n_2-1,\ \delta_2(p)}(l_3)\left(F_{n_1-1,\ \delta_1(p)}(l_2) - F_{n_1-1,\ \delta_1(p)}(l_1)\right) \tag{16}
$$

and

$$
N_\phi(p) = N(p) = n_1 + n_2\left(F_{n_1-1,\ \delta_1(p)}(l_2) - F_{n_1-1,\ \delta_1(p)}(l_1)\right), \tag{17}
$$

respectively, where $\delta_i(p) = \sqrt{n_i}\,\Phi^{-1}(p)$ for $i = 1, 2$.

ϕ^* is computed by

$$
\begin{aligned}
&\text{(i)} \quad L_\phi(p_1) \geq 1 - \alpha \\
&\text{(ii)} \quad L_\phi(p_2) \leq \beta \\
&\text{(iii)} \quad N_{max}(\phi^*) = \min_{\phi \in Z} N_{max}(\phi),
\end{aligned}
\tag{18}
$$

where Z denotes the set of the double sampling plans ϕ fulfilling (18)(i) and (ii). The algorithm for the computation of ϕ^* is similar to that developed by VANGJELI (2009) for computing AM-double stage tests. The parameters of the approximated AM-double sampling plan $\widetilde{\lambda} = \begin{pmatrix} \tilde{n}_1 & \tilde{k}_1 & \tilde{k}_2 \\ \tilde{n}_2 & \tilde{k}_3 \end{pmatrix}$ in the two-sided case are given by $\tilde{n}_1 = n_1$, $\tilde{n}_2 = n_2$ for both procedures and

$$
\tilde{k}_1 = \Phi\left(\frac{l_1}{\sqrt{n_1}}\right), \ \tilde{k}_2 = \Phi\left(\frac{l_2}{\sqrt{n_1}}\right), \ \tilde{k}_3 = \Phi\left(\frac{l_3}{\sqrt{n_2}}\right),
$$

for the first procedure, and

$$
\tilde{k}_1 = B\left(\frac{1}{2} + \frac{l_1}{2(n_1 - 1)}\right), \ \tilde{k}_2 = B\left(\frac{1}{2} + \frac{l_2}{2(n_1 - 1)}\right), \ \tilde{k}_3 = B\left(\frac{1}{2} + \frac{l_3}{2(n_2 - 1)}\right),
$$

for the second procedure. Identically to the computation of the single sampling plans, we set

$$
\mu_0 = \frac{L + U}{2},
$$

$$
\sigma_0(p) = \frac{L - U}{2\Phi^{-1}\left(\frac{p}{2}\right)}
$$

and define for given p $(0 < p < 1)$ and σ $(0 < \sigma \leq \sigma_0(p))$, $\mu = \mu(\sigma, p)$ by inverting (1) with $\mu \geq \mu_0$. Setting

$$
L_{\lambda_{AM}}(\sigma; p) := L_{\lambda_{AM}}(\mu(\sigma, p), \sigma),
\tag{19}
$$

$$
N_{\lambda_{AM}}(\sigma; p) := N_{\lambda_{AM}}(\mu(\sigma, p), \sigma),
\tag{20}
$$

the AM-double sampling plan λ_{AM} is given by the condition

$$
\begin{aligned}
&\text{(i)} \quad \min_{0<\sigma\leq\sigma_0(p)} L_{\lambda_{AM}}(\sigma; p_1) \geq 1 - \alpha \\
&\text{(ii)} \quad \max_{0<\sigma\leq\sigma_0(p)} L_{\lambda_{AM}}(\sigma; p_2) \leq \beta \quad \text{(Figure 1)} \\
&\text{(iii)} \quad N_{max}(\phi^*) = \min_{\phi\in Z} N_{max}(\phi),
\end{aligned}
\tag{21}
$$

the ASN maximum of λ_{AM} is defined

$$
N_{max}(\lambda_{AM}) = \max_{p,\ \sigma} N_{\lambda_{AM}}(\sigma; p) \quad \text{(Figure 2)}.
\tag{22}
$$

λ_{AM} is calculated as follows:

1. Compute the single sampling plan $(n,\ k)$ and save α^*, β^*

2. Starting from $\alpha^{**} = \alpha^*, \beta^{**} = \beta^*$, compute $\widetilde{\lambda}$ for $p_1, p_2, \alpha^{**}, \beta^{**}$, verify (21)(i), (ii) and vary α^{**}, β^{**} until (21)(i), (ii) are true.

Example 1

For $L = 1$, $U = 9$, $p_1 = 0.01$, $p_2 = 0.06$, $\alpha = \beta = 0.1$, we get

(i) $(n^*,\ k^*) = (36,\ 0.02645943143)$ and $\alpha^* = 0.082$, $\beta^* = 0.1$.

α^{**}	β^{**}	$\widetilde{\lambda}$		$N_{max}(\phi^*)$	$\min\limits_{\sigma} L_{\widetilde{\lambda}}(\sigma; p_1)$	$\max\limits_{\sigma} L_{\widetilde{\lambda}}(\sigma; p_2)$
0.082	0.1	25 0.016988 0.034411	19 0.029059	31.31538	0.8882933643	0.0999999893
0.076	0.1	26 0.017688 0.034554	19 0.029215	32.16417	0.8960633610	0.0999999882
0.072	0.1	26 0.017577 0.035291	20 0.029275	32.75441	0.9010124424	0.0999999889

where $\lambda^*_{AM} = \begin{pmatrix} 26 & 0.017577 & 0.035291 \\ 20 & 0.029275 & \end{pmatrix}$ with $N_{max}(\lambda^*_{AM}) = 32.75439$.

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(ii) $(\hat{n},\ \hat{k}) = (34,\ 0.02262119182)$ and $\alpha^* = 0.098,\ \beta^* = 0.094$.

α^{**}	β^{**}	$\widetilde{\lambda}$			$N_{max}(\phi^*)$	$\min\limits_{\sigma} L_{\widetilde{\lambda}}(\sigma; p_1)$	$\max\limits_{\sigma} L_{\widetilde{\lambda}}(\sigma; p_2)$
0.098	0.094	24	0.011840	0.029264	30.19915	0.9002805676	0.1004854552
		18	0.023800				
0.098	0.093	24	0.012148	0.029093	30.34629	0.9002364848	0.0995258042
		19	0.023424				

where $\widehat{\lambda}_{AM} = \begin{pmatrix} 24 & 0.012148 & 0.029093 \\ 19 & 0.023424 & \end{pmatrix}$ with $N_{max}(\widehat{\lambda}_{AM}) = 30.34628$.

Example 2

For $\ L = 1,\ U = 9,\ p_1 = 0.01,\ p_2 = 0.03,\ \alpha = \beta = 0.1$, we get

(i) $(n^*,\ k^*) = (115,\ 0.0178762881)$ and $\alpha^* = 0.085,\ \beta^* = 0.1$.

α^{**}	β^{**}	$\widetilde{\lambda}$			$N_{max}(\phi^*)$	$\min\limits_{\sigma} L_{\widetilde{\lambda}}(\sigma; p_1)$	$\max\limits_{\sigma} L_{\widetilde{\lambda}}(\sigma; p_2)$
0.085	0.1	79	0.013777	0.021642	101.1604	0.8948821204	0.0999999568
		64	0.018624				
0.083	0.1	80	0.013902	0.021726	102.0913	0.8972199027	0.0999999565
		64	0.018464				
0.080	0.1	81	0.014029	0.021742	103.5434	0.9008045948	0.0999999565
		66	0.018537				

where $\lambda^*_{AM} = \begin{pmatrix} 81 & 0.014029 & 0.021742 \\ 66 & 0.018537 & \end{pmatrix}$ with $N_{max}(\lambda^*_{AM}) = 103.5434$.

(ii) $(\hat{n},\ \hat{k}) = (113,\ 0.01678745123)$ and $\alpha^* = 0.096,\ \beta^* = 0.094$.

α^{**}	β^{**}	$\widetilde{\lambda}$			$N_{max}(\phi^*)$	$\min\limits_{\sigma} L_{\widetilde{\lambda}}(\sigma; p_1)$	$\max\limits_{\sigma} L_{\widetilde{\lambda}}(\sigma; p_2)$
0.096	0.094	78	0.012471	0.020036	99.13899	0.9000170882	0.1014154024
		62	0.017078				
0.096	0.092	78	0.012406	0.020069	100.1070	0.9000091667	0.0993767725
		64	0.016981				

where $\widehat{\lambda}_{AM} = \begin{pmatrix} 78 & 0.012406 & 0.020069 \\ 64 & 0.016981 & \end{pmatrix}$ with $N_{max}(\widehat{\lambda}_{AM}) = 100.1070$.

Remark: Numerical investigations have shown:

(i) There are nonessential differences between $N_{max}(\phi^*)$ and $N_{max}(\lambda_{AM})$.

(ii) Since the parameters α^{**} and β^{**} vary by a constant of 0.001, we cannot presume that the calculated λ_{AM} is the precise AM-double sampling plan. Finding plans that fulfill more sharply (21) (i), (ii) and feature an insignificantly lower N_{max} would require an unreasonable amount of time.

(iii) For the calculated double sampling plans, it holds that

$$N_{max}(\widehat{\lambda}_{AM}) < N_{max}(\lambda_{AM}^*)$$

(Figure 3), which is consistent with the known superiority of $\hat{p}$ over p^* from the single sampling literature.
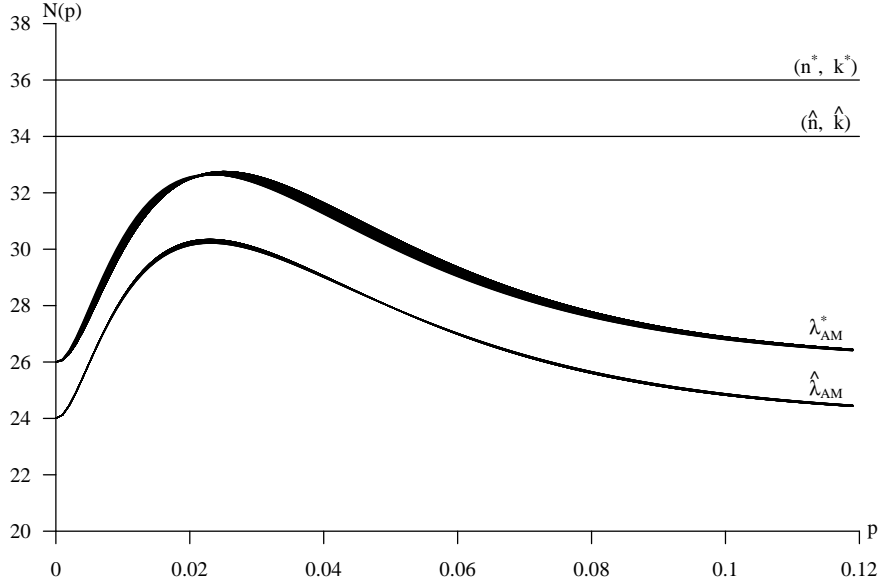

Figure 3: ASN bands for λ_{AM}^* and $\widehat{\lambda}_{AM}$ defined by $p_1 = 0.01$, $p_2 = 0.06$ and $\alpha = \beta = 0.1$

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